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Ruled surfaces with pointwise 1-type Gauss map

Young Ho Kim*, Dae Won Yoon
Department of Mathematics, Teachers College, Kyungpook National University, Taegu 702-701, South Korea

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Abstract

In this paper, we study ruled surfaces in a three-dimensional Minkowski space with pointwise 1-type Gauss map and obtain the complete classification theorems for those. We also obtain a new characterization of minimal ruled surfaces in a three-dimensional Minkowski space. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Geometers have been interested in studying minimal surfaces for a long time. In particular, the only minimal ruled surfaces in a three-dimensional Euclidean space $\mathbb{E}^3$ are the planes and the helicoids. In 1983, Kobayashi [15] classified space-like ruled minimal surface in a three-dimensional Minkowski space $\mathbb{E}^3_1$, and de Woestijne [16] extended it to the Lorentz version in 1988.

In late 1970s Chen [5,6] introduced the notion of Euclidean immersions of finite type. Essentially these are submanifolds whose immersion into $\mathbb{E}^m$ is constructed by making use of a finite number of $\mathbb{E}^m$-valued eigenfunctions of their Laplacian. The first results on this subject have been collected in the book [6]; for a recent survey, see [7]. Many works were done to characterize or classify submanifolds in terms of finite type. In a framework of the theory of finite type, Chen and Piccinni [8] made a general study on submanifolds of Euclidean spaces with finite type Gauss map and classified compact surfaces of 1-type Gauss map. Several geometers also studied submanifolds of Euclidean spaces or pseudo-Euclidean spaces with finite type Gauss map (cf. [1,3,4,9,14], etc.). On the other hand, Baikoussis and

* Corresponding author.
E-mail address: yhkim@kyungpook.ac.kr (Y.H. Kim)

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Blair [2] studied ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map G satisfies a special condition,

$$\Delta G = AG, \qquad A \in \mathrm{Mat}(3, \mathbb{R}), \tag{1.1}$$

where Δ denotes the Laplacian of the surface with respect to the induced metric and $\mathrm{Mat}(3, \mathbb{R})$ is the set of 3×3-real matrices. Also, for the Lorentz version Choi [9] investigated ruled surfaces with non-null base curve satisfying the condition (1.1) in a three-dimensional Minkowski space $\mathbb{E}^3_1$. Furthermore, Alías et al., [1] studied ruled surfaces in a three-dimensional Minkowski space $\mathbb{E}^3_1$ with null rulings satisfying the condition (1.1). In such cases, it is well known that all surfaces obtained in [1,2,9] satisfy the condition $\Delta G = \lambda G, \quad \lambda \in \mathbb{R}$.

However, there may be some other ruled surfaces satisfying $\Delta G = fG$ for some smooth function f, for example, see Section 3. Relating with such matters, we raise the following problem:

Problem. *Classify all submanifolds in an m-dimensional Euclidean space* $\mathbb{E}^m$ (*or Minkowski space* $\mathbb{E}^m_1$) *satisfying the condition*

$$\Delta G = fG \tag{1.2}$$

for some function f.

A submanifold M in $\mathbb{E}^m$ (or $\mathbb{E}^m_1$) is said to be of *pointwise 1-type Gauss map* if it satisfies (1.2).

For the above problem, Choi and Kim [10] recently proved the following theorem:

Theorem A. *The ruled surfaces in* $\mathbb{E}^3$ *with pointwise 1-type Gauss map are an open portion of the plane, the circular cylinder and the helicoid.*

In this article, we investigate the Lorentz version of the above theorem, and give a complete classification of ruled surfaces with pointwise 1-type Gauss map.

On the other hand, Kim et al. [13] completely classified ruled surfaces in $\mathbb{E}^m_1$ with 1-type Gauss map.

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless otherwise mentioned.

2. Preliminaries

An m–dimensional vector space $L = L^m_1$ with scalar product $\langle , \rangle$ of index 1 is called a *Lorentz vector space*. In particular, if $L = \mathbb{E}^m_1$, $m \geq 2$, it is called a *Minkowski m-space*. A vector X of L^m_1 is said to be *space-like* if $\langle X, X \rangle > 0$ or $X = 0$, *time-like* if $\langle X, X \rangle < 0$ and *light-like or null* if $\langle X, X \rangle = 0$ and $X \neq 0$. A time-like or light-like vector in L^m_1 is said to be *causal*. For the Lorentz vector space the next two lemmas are well known and useful (See [12]).

Lemma 2.1. *There are no causal vectors in L_1^m orthogonal to a time-like vector.*

Lemma 2.2. *Two light-like vectors are orthogonal if and only if they are linearly dependent.*

Let $X = (X_i)$ and $Y = (Y_i)$ be the vectors in a three-dimensional Lorentz vector space L_1^3, then the scalar product of X and Y is defined by

$$\langle X, Y\rangle = -X_1Y_1 + X_2Y_2 + X_3Y_3, \tag{2.1}$$

which is called *a Lorentz product*. Furthermore, a Lorentz cross product $X \times Y$ is given by

$$X \times Y = (-X_2Y_3 + X_3Y_2, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1). \tag{2.2}$$

Then it is easily seen that the Lorentz cross product satisfies the following.

Lemma 2.3. *For vector fields X,Y,Z and W in L_1^3,*

$$X \times Y = 0 \Leftrightarrow X \text{ and } Y \text{ are linearly dependent}. \tag{2.3}$$

$$X \times Y = -Y \times X. \tag{2.4}$$

$$\langle X \times Y, Z\rangle = \langle Y \times Z, X\rangle. \tag{2.5}$$

$$\langle X \times Y, X\rangle = \langle X \times Y, Y\rangle = 0. \tag{2.6}$$

$$\langle X \times Y, Z \times W\rangle = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle. \tag{2.7}$$

Let M be a pseudo-Riemannian surface in a three-dimensional Minkowski space $\mathbb{E}_1^3$. The map $G : M \to Q^2(\varepsilon) \subset \mathbb{E}_1^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of surface M, where $\varepsilon(= \pm 1)$ denotes the sign of the vector field G and $Q^2(\varepsilon)$ is a 2-dimensional space form as follows:

$$Q^2(\varepsilon) = \begin{cases} S_1^2(1) & \text{in } \mathbb{E}_1^3, \quad \text{if } \varepsilon = 1; \\ H^2(-1) & \text{in } \mathbb{E}_1^3, \quad \text{if } \varepsilon = -1. \end{cases}$$

It is well known that in terms of local coordinates $\{x_i\}$ of M the Laplacian can be written as

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}}\sum_{i,j}\frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{G}|}g^{ij}\frac{\partial}{\partial x^j}\right), \tag{2.8}$$

where $\mathcal{G} = \det(g_{ij})$, $(g^{ij}) = (g_{ij})^{-1}$ and (g_{ij}) are the components of the metric of M with respect to $\{x_i\}$.

Now, we define a ruled surface M in a three-dimensional Minkowski space $\mathbb{E}_1^3$. Let I be an open interval in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{E}_1^3$ defined on I and $\beta = \beta(s)$ a transversal vector field along α. For an open interval J of $\mathbb{R}$ we have the parametrization for M

$$x = x(s, t) = \alpha(s) + t\beta(s), \qquad s \in I, \quad t \in J. \tag{2.9}$$

The curve $\alpha = \alpha(s)$ is called a *base curve* and $\beta = \beta(s)$ a *director curve*. In particular, if β is constant, the ruled surface is said to be *cylindrical*, and *non-cylindrical* otherwise.

First of all, we consider that the base curve α is space-like or time-like. In that case, the director curve β can be naturally chosen so that it is orthogonal to α. Furthermore, we have ruled surfaces of five different kinds according to the character of the base curve α and the director curve β as follows: If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. When β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or light-like, respectively. When β is time-like, β' must be space-like according to Lemma 2.1. In this case, M said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, it is also said to be of type M_-^1 or M_-^2 if β' is non-null or light-like, respectively. Note that in the case of type M_- the director curve β is always space-like (cf. [9,14]). The ruled surface of type M_+^1 or M_+^2 (resp. M_+^3, M_-^1 or M_-^2) is clearly space-like (resp. time-like).

But, if the base curve α is a light-like curve and the vector field β along α is a light-like vector field, then the ruled surface M is called a *null scroll*. In particular, a null scroll with Cartan frame is said to be a *B-scroll* [11]. It is also a time-like surface.

3. Some examples

Before going into the study of ruled surfaces with the condition $\Delta G = fG$, let us see some examples of surfaces in $\mathbb{E}_1^3$ satisfying that condition. They will be parts of our classifications of ruled surfaces.

Example 3.1 (Helicoid of the 1st kind). For constants a and b with $|a| > |b| > 0$, we consider the surface M in $\mathbb{E}_1^3$ defined by

$$x(s,t) = (-bs, (t+a)\cos s, (t+a)\sin s),$$

where $t < \min(-a-b, -a+b)$ or $t > \max(-a-b, -a+b)$.

This parametrization defines a non-cylindrical ruled surface of type M_+^1 in $\mathbb{E}_1^3$, which is called a helicoid of the 1st kind as space-like surface. In this case, the Gauss map G is given by

$$G = \frac{1}{\sqrt{(t+a)^2 - b^2}}(t+a, b\sin s, -b\cos s).$$

The Laplacian ΔG of the Gauss map G is obtained as

$$\Delta G = \frac{-2b^2}{((t+a)^2 - b^2)^2} G.$$

Example 3.2 (Helicoid of the 2nd kind). For constants a and b with $|b| > |a|$, we consider the surface M in $\mathbb{E}_1^3$ defined by

$$x(s,t) = ((t+a)\sinh s, (t+a)\cosh s, -bs),$$

where $\min(-a-b, -a+b) < t < \max(-a-b, -a+b)$.

This parametrization defines a non-cylindrical ruled surface of type M_+^1 in $\mathbb{E}_1^3$, which is called a helicoid of the 2nd kind as space-like surface. The Gauss map G for it is given by

$$G = \frac{1}{\sqrt{b^2-(t+a)^2}}(-b\cosh s, -b\sinh s, t+a)$$

and the Laplacian ΔG of the Gauss map G is derived as

$$\Delta G = \frac{-2b^2}{(b^2-(t+a)^2)^2}G.$$

Example 3.3 (Conjugate of Enneper's surface of the 2nd kind). The surface in $\mathbb{E}_1^3$ defined by

$$x(s,t) = \left(\frac{1}{6}s^3 + ts, -\frac{1}{6}s^3 - ts + s, \frac{1}{2}s^2 + t\right)$$

is a non-cylindrical ruled surface of type M_+^2, which is said to be a conjugate of Enneper's surface of the 2nd kind as space-like surface. The Gauss map G is obtained by

$$G = \frac{1}{\sqrt{-2t+1}}\left(-\frac{1}{2}s^2 + t - 1, \frac{1}{2}s^2 - t, -s\right).$$

The Laplacian ΔG of the Gauss map G can be expressed as

$$\Delta G = \frac{-2}{(-2t+1)^2}G, \qquad t < \frac{1}{2}.$$

Example 3.4 (Helicoid of the 1st kind). For constants a and b satisfying $|a| < |b|$, we consider the surface M in $\mathbb{E}_1^3$ defined by

$$x(s,t) = (-bs, (t+a)\cos s, (t+a)\sin s),$$

where $\min(-a-b, -a+b) < t < \max(-a-b, -a+b)$.

This parametrization defines a non-cylindrical ruled surface of type M_-^1 in $\mathbb{E}_1^3$, which is called a helicoid of the 1st kind as time-like surface. Similarly to Example 3.1, we can get the Laplacian ΔG of the Gauss map G

$$\Delta G = \frac{-2b^2}{(b^2-(t+a)^2)^2}G.$$

Example 3.5 (Helicoid of the 2nd kind). For constants a and b with $|a| > |b| > 0$, we consider the surface M in $\mathbb{E}_1^3$ defined by

$$x(s,t) = ((t+a)\sinh s, (t+a)\cosh s, -bs),$$

where $t < \min(-a-b, -a+b)$ or $t > \max(-a-b, -a+b)$.

This parametrization defines a non-cylindrical ruled surface of type M_-^1 in $\mathbb{E}_1^3$, which is said to be a helicoid of the 2nd kind as time-like surface. The Laplacian ΔG of the Gauss map G is determined by

$$\Delta G = \frac{-2b^2}{((t+a)^2 - b^2)^2} G.$$

Example 3.6 (Helicoid of the 3rd kind). For constants a and b satisfying $|a| < |b|$, we consider the surface M in $\mathbb{E}_1^3$ defined by

$$x(s,t) = ((t+a)\cosh s, bs, (t+a)\sinh s),$$

where $\min(-a-b, -a+b) < t < \max(-a-b, -a+b)$.

This parametrization defines a non-cylindrical ruled surface of type M_+^3 in $\mathbb{E}_1^3$, which is called a helicoid of the 3rd kind as time-like surface. In this case, the Laplacian ΔG of the Gauss map G can be expressed as

$$\Delta G = \frac{2b^2}{(b^2 - (t+a)^2)^2} G.$$

Example 3.7 (Conjugate of Enneper's surface of the 2nd kind). The surface in $\mathbb{E}_1^3$ defined by

$$x(s,t) = \left(\frac{1}{6}s^3 + ts + s, -\frac{1}{6}s^3 - ts, \frac{1}{2}s^2 + t\right)$$

is a non-cylindrical ruled surface of type M_-^2, which is said to be a conjugate of Enneper's surface of the 2nd kind as time-like surface. The Laplacian ΔG of the Gauss map G is given by

$$\Delta G = \frac{-2}{(2t+1)^2} G, \qquad t > -\frac{1}{2}.$$

Example 3.8 (B-scroll, cf. [1]). Let $\gamma = \gamma(s)$ be a light-like curve in $\mathbb{E}_1^3$ with Cartan frame $\{A, B, C\}$, i.e., A, B, C are vector fields along γ in $\mathbb{E}_1^3$ satisfying the following conditions:

$$\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = 1, \quad \langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,$$
$$\gamma' = A, \quad C' = -aA - k(s)B,$$

a being a constant and $k(s)$ a function vanishing nowhere.

Let $x = x(s,t) = \gamma(s) + tB(s)$. Then, it is a time-like surface in $\mathbb{E}_1^3$, which is called a *B-scroll* [11]. The Gauss map G is given by

$$G(s,t) = -atB(s) + C(s).$$

As for the shape operator S we have that

$$G_s := \frac{\partial G}{\partial s} = -ax_s - k(s)x_t, \quad G_t := \frac{\partial G}{\partial t} = -ax_t.$$

So S is written down, relative to the usual frame $\{x_s, x_t\}$, as

$$\begin{pmatrix} a & 0 \\ k(s) & a \end{pmatrix}.$$

Thus, it has constant mean curvature $\alpha = a$ and constant Gaussian curvature $K = a^2$. Furthermore, the Laplacian ΔG of the Gauss map G can be expressed as

$$\Delta G = \lambda G, \quad \lambda = -2a^2.$$

On the other hand, a B-scroll is minimal if and only if it is flat.

Remarks.
1. All surfaces given in the above examples are congruence to surfaces described in [16].
2. There are surfaces satisfying the condition (1.2) as cylindrical ruled surfaces. For example, (1) a non-degenerated plane, (2) a hyperbolic cylinder $\mathbb{H}^1 \times \mathbb{R}$, (3) a Lorentz circular cylinder $\mathbb{S}^1_1 \times \mathbb{R}$, (4) a circular cylinder $\mathbb{R}^1_1 \times \mathbb{S}^1$ of index 1(for details, see [9]).

4. Classification theorems

In this section, we will classify the ruled surfaces in terms of pointwise 1-type Gauss map.

Suppose that the ruled surfaces M satisfy the condition (1.2). Then, the tangential component of ΔG vanishes, i.e.,

$$\Delta G - \varepsilon \langle \Delta G, G \rangle G = 0. \tag{4.1}$$

We divide ruled surfaces in $\mathbb{E}^3_1$ into three typical types according to the character of the base curve α and the vector field β, i.e., cylindrical ruled surfaces, non-cylindrical ruled surfaces and null scrolls.

Theorem 4.1. *The only cylindrical ruled surfaces with space-like or time-like base curve in a three-dimensional Minkowski space with pointwise 1-type Gauss map are an open part of one of the following surfaces:*
1. *a Euclidean plane,*
2. *a Minkowski plane,*
3. *the hyperbolic cylinder,*
4. *the Lorentz circular cylinder,*
5. *the circular cylinder of index* 1.

Proof. Let M be a cylindrical ruled surface in $\mathbb{E}^3_1$, i.e., $\alpha = \alpha(s)$ is a space-like or time-like smooth curve and $\beta = \beta(s)$ a space-like or time-like unit constant vector field along α or orthogonal to α and s the arc-length of α. Then, M is parametrized by

$$x = x(s, t) = \alpha(s) + t\beta$$

such that $\langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1)$, $\langle \alpha', \beta \rangle = 0$, $\langle \beta, \beta \rangle = \varepsilon_2 (= \pm 1)$.

Also the cylindrical ruled surface M is only of type M^1_+, M^3_+ or M^1_-.

In order to prove the theorem, we split it into two cases.

Case 1: Let M be a cylindrical ruled surface of type M^1_+ or M^1_-, i.e., $\varepsilon_2 = 1$. Performing a Lorentz transformation, we may assume that $\beta = (0, 0, 1)$ without loss of generality. Then α may be regarded as the plane curve $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$. The Gauss map of M is defined by $G = \alpha' \times \beta = (-\alpha_2', -\alpha_1', 0)$, where the prime denotes the derivative with respect to s. Since the induced pseudo-Riemannain metric is given by $\langle x_s, x_s\rangle = \varepsilon_1$, $\langle x_s, x_t\rangle = 0$ and $\langle x_t, x_t\rangle = 1$, the Laplacian ΔG of the Gauss map G is obtained by $\Delta G = (\varepsilon_1\alpha_2''', \varepsilon_1\alpha_1''', 0)$. Then, (1.2) implies that we have the following system of differential equations:

$$\varepsilon_1\alpha_2'''(s) = -f(s,t)\alpha_2'(s), \qquad \varepsilon_1\alpha_1'''(s) = -f(s,t)\alpha_1'(s). \tag{4.2}$$

From this, we see that f is a function of s only. In order to solve the above equations we first consider the surface M of type M^1_+, i.e., $\varepsilon_1 = 1$. So we get $\langle\alpha', \alpha'\rangle = -\alpha_1'^2 + \alpha_2'^2 = 1$. Accordingly, we may put α_1' and α_2' as follows:

$$\alpha_1' = \sinh\theta, \qquad \alpha_2' = \cosh\theta,$$

where $\theta = \theta(s)$. Putting these into (4.2), we have

$$\theta''\sinh\theta + (\theta'^2 + f(s,t))\cosh\theta = 0, \qquad (\theta'^2 + f(s,t))\sinh\theta + \theta''\cosh\theta = 0,$$

which implies

$$\theta'' = 0, \qquad f(s,t) = -\theta'^2.$$

Therefore, f is a constant. Using Proposition 3.1 of [9], we conclude that M is an open portion of a Euclidean plane and the hyperbolic cylinder.

Next, we are concerned with the ruled surface M of type M^1_-, i.e., $\varepsilon_1 = -1$. Since $\langle\alpha', \alpha'\rangle = -\alpha_1'^2 + \alpha_2'^2 = -1$, we may put

$$\alpha_1' = \cosh\theta, \qquad \alpha_2' = \sinh\theta$$

where $\theta = \theta(s)$. By the similar discussion as above, we can get

$$(\theta'^2 - f(s,t))\sinh\theta + \theta''\cosh\theta = 0, \qquad \theta''\sinh\theta + (\theta'^2 - f(s,t))\cosh\theta = 0,$$

from which,

$$\theta'' = 0, \qquad f(s,t) = \theta'^2.$$

Thus, f is also a constant. It shows that M is an open portion of a Minkowski plane and the Lorentz circular cylinder according to Proposition 3.1 of [9].

Case 2: Let M be a cylindrical ruled surface of type M^3_+, i.e., $\varepsilon_1 = 1, \varepsilon_2 = -1$. As in the previous case, by an appropriate rigid motion, we may assume $\beta = (1, 0, 0)$ and $\alpha(s) = (0, \alpha_2(s), \alpha_3(s))$ without loss of generality. The Gauss map G of M is given by $G = (0, \alpha_3', -\alpha_2')$ and the Laplacian ΔG of the Gauss map G is obtained by $\Delta G = (0, -\alpha_3''', \alpha_2''')$. Furthermore, the condition (1.2) implies

$$\alpha_3''' = -f(s,t)\alpha_3', \qquad \alpha_2''' = -f(s,t)\alpha_2' \tag{4.3}$$

Since $\langle \alpha', \alpha' \rangle = \alpha_2'^2 + \alpha_3'^2 = 1$, we may put

$$\alpha_2' = \cos\theta, \qquad \alpha_3' = \sin\theta$$

where $\theta = \theta(s)$. Similarly to Case 1, we can obtain that f is a constant. Thus, M is an open part of a Minkowski plane and the circular cylinder of index 1 according to Proposition 3.1 of [9]. □

Theorem 4.2. *Let M be a non-cylindrical ruled surface with space-like or time-like base curve in a three-dimensional Minkowski space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following surfaces:*
1. *the helicoid of the* 1*st kind as space-like or time-like surface,*
2. *the helicoid of the* 2*nd kind as space-like or time-like surface,*
3. *the helicoid of the* 3*rd kind as space-like or time-like surface,*
4. *the conjugate of Enneper's surfaces of the* 2*nd kind as space-like or time-like surface.*

Proof. We consider two cases separately.

Case 1: Let M be a non-cylindrical ruled surface of one of the three types M_+^1, M_+^3 or M_-^1 according to the character of the base curve α and the director curve β.
1. $\alpha = \alpha(s)$ is space-like and $\beta = \beta(s)$ is space-like,
2. $\alpha = \alpha(s)$ is space-like and $\beta = \beta(s)$ is time-like,
3. $\alpha = \alpha(s)$ is time-like and $\beta = \beta(s)$ is space-like,

where s is the arc-length of the director curve β.

We also express the ruled surface M is parametrized by, up to a rigid motion,

$$x = x(s,t) = \alpha(s) + t\beta(s) \tag{4.4}$$

such that $\langle \alpha', \beta \rangle = 0$, $\langle \beta, \beta \rangle = \varepsilon_2 (= \pm 1)$ and $\langle \beta', \beta' \rangle = \varepsilon_3 (= \pm 1)$. And we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. For later use, we define smooth functions q, u and v as follows:

$$q = \|x_s\|^2 = \varepsilon_4 \langle x_s, x_s \rangle, \quad u = \langle \alpha', \beta' \rangle, \quad v = \langle \alpha', \alpha' \rangle, \tag{4.5}$$

where $\varepsilon_4 (= \pm 1)$ is the sign of the vector x_s. Then, the induced pseudo-Riemannain metric on M is obtained by $\langle x_s, x_s \rangle = \varepsilon_4 q$, $\langle x_s, x_t \rangle = 0$ and $\langle x_t, x_t \rangle = \varepsilon_2$. If we make use of (2.8) together with such functions q, u and v, the Laplacian Δ of M can be expressed as follows [14]:

$$\Delta = -\varepsilon_4 \left(\frac{1}{q} \frac{\partial^2}{\partial s^2} - \frac{1}{2} \frac{1}{q^2} \frac{\partial q}{\partial s} \frac{\partial}{\partial s} \right) - \varepsilon_2 \left(\frac{\partial^2}{\partial t^2} + \frac{1}{2} \frac{1}{q} \frac{\partial q}{\partial t} \frac{\partial}{\partial t} \right). \tag{4.6}$$

Furthermore, the Gauss map G of M is obtained by

$$G = \left(\frac{1}{\|x_s \times x_t\|} \right) x_s \times x_t = q^{-1/2}(A + tB) \tag{4.7}$$

and the smooth function q is given by

$$q = \varepsilon_4 (\varepsilon_3 t^2 + 2ut + v), \tag{4.8}$$

where we put $A = \alpha' \times \beta$ and $B = \beta' \times \beta$. By a straightforward computation, the Laplacain ΔG of the Gauss map G with the help of (4.7) turns out to be

$$\Delta G = \{2\varepsilon_2\varepsilon_3\varepsilon_4 q^{-1} - 2\varepsilon_2(\varepsilon_3 t + u)^2 q^{-2} - \varepsilon_4(2u't + v')^2 q^{-3} + \tfrac{1}{2}(2u''t + v'')q^{-2}\}G + \tfrac{1}{2}q^{-5/2}\{2\varepsilon_2\varepsilon_4 q(-\varepsilon_3 A + uB) - 2\varepsilon_4 q(A'' + tB'') + 3(2u't + v')(A' + tB')\}. \tag{4.9}$$

The direct computation of the left-hand side of (4.1) gives a polynomial in t with functions of s as the coefficients by adjusting the power of the function q and thus they must be zero. we have then

$$B'' - \varepsilon\varepsilon_3\varepsilon_4 < B'', B > B = 0, \tag{4.10}$$

$$A'' + (\varepsilon_2\varepsilon_3 - \varepsilon\varepsilon_3\varepsilon_4\langle B'', B\rangle)A + 4\varepsilon_3 uB'' - 3\varepsilon\varepsilon_3 u'B' - (\varepsilon_2 u + \varepsilon\varepsilon_3\varepsilon_4\langle A'', B\rangle + \varepsilon\varepsilon_3\varepsilon_4\langle A, B''\rangle + 2\varepsilon\varepsilon_4 u\langle B, B''\rangle)B = 0, \tag{4.11}$$

$$8\varepsilon_3\varepsilon_4 uA'' - 6\varepsilon\varepsilon_3\varepsilon_4 u'A' + (8\varepsilon_2\varepsilon_4 u - 2\varepsilon\varepsilon_3\langle A'', B\rangle - 2\varepsilon\varepsilon_3\langle A, B''\rangle - 4\varepsilon u\langle B, B''\rangle)A + (4\varepsilon_3\varepsilon_4 v + 8\varepsilon_4 u^2)B'' - \varepsilon(12\varepsilon_4 uu' + 3\varepsilon_3\varepsilon_4 v')B' - (8\varepsilon_2\varepsilon_3 u^2 + 2\varepsilon\varepsilon_3 u^2 - 2\varepsilon v + 6\varepsilon\varepsilon_2 u'^2 + 2\varepsilon\varepsilon_3\langle A, A''\rangle + 4\varepsilon u\langle A'', B\rangle + 4\varepsilon u\langle A, B''\rangle + 2\varepsilon v\langle B'', B\rangle)B = 0, \tag{4.12}$$

$$\varepsilon_4(4\varepsilon_3 v + 8u^2)A'' - \varepsilon\varepsilon_4(12uu' + 3\varepsilon_3 v')A' + (4\varepsilon_2\varepsilon_4 v + 8\varepsilon_2\varepsilon_3\varepsilon_4 u^2 + 2\varepsilon v - 2\varepsilon\varepsilon_3 u^2 - 6\varepsilon\varepsilon_2 u'^2 - 2\varepsilon\varepsilon_3\langle A, A''\rangle - 4\varepsilon u\langle A'', B\rangle - 4\varepsilon u\langle A, B''\rangle - 2\varepsilon v\langle B'', B\rangle)A + 8\varepsilon_4 uvB'' - \varepsilon\varepsilon_4(6u'v + 6uv')B' - (4\varepsilon_2\varepsilon_3\varepsilon_4 uv + 8\varepsilon_2\varepsilon_4 u^3 - 4\varepsilon\varepsilon_3 uv + 4\varepsilon u^3 + 6\varepsilon\varepsilon_2 u'v' + 4\varepsilon u\langle A, A''\rangle + 2\varepsilon v\langle A'', B\rangle + 2\varepsilon v\langle A, B''\rangle)B = 0, \tag{4.13}$$

$$16\varepsilon_4 uvA'' - \varepsilon\varepsilon_4(12u'v + 12uv')A' + (16\varepsilon_2\varepsilon_3\varepsilon_4 uv + 8\varepsilon\varepsilon_3 uv - 8\varepsilon u^3 - 12\varepsilon\varepsilon_2 u'v' - 8\varepsilon u\langle A, A''\rangle - 4\varepsilon v\langle A'', B\rangle - 4\varepsilon v\langle A, B''\rangle)A + 4\varepsilon_4 v^2 B'' - 6\varepsilon\varepsilon_4 vv'B' - (16\varepsilon_2\varepsilon_4 u^2 v - 4\varepsilon\varepsilon_3 v^2 + 4\varepsilon u^2 v + 3\varepsilon\varepsilon_2 v'^2 + 4\varepsilon v\langle A, A''\rangle)B = 0, \tag{4.14}$$

$$4\varepsilon_4 v^2 A'' - 6\varepsilon\varepsilon_4 vv'A' + (4\varepsilon_2\varepsilon_3\varepsilon_4 v^2 + 4\varepsilon\varepsilon_3 v^2 - 4\varepsilon u^2 v - 3\varepsilon\varepsilon_2 v'2 - 4\varepsilon v\langle A, A''\rangle)A - 4\varepsilon_2\varepsilon_4 uv^2 B = 0. \tag{4.15}$$

It follows from (4.10) that

$$\langle B'', B'\rangle = 0, \tag{4.16}$$

i.e., $\langle B', B'\rangle = c$ for some constant c. This implies

$$\langle B'', B\rangle = -c. \tag{4.17}$$

Thus, (4.10) can be rewritten in the form

$$B'' = -\varepsilon\varepsilon_3\varepsilon_4 cB, \tag{4.18}$$

which implies

$$\langle A, B''\rangle = \varepsilon\varepsilon_2\varepsilon_3\varepsilon_4 cu. \tag{4.19}$$

Consequently, using (4.16),(4.17),(4.18) and (4.19) we can eliminate A'', A' and B' so that

$$(v'^2 - 4\varepsilon_3 u'^2 v)A + (8uu'^2 v - 4\varepsilon_3 u' v v')B = 0, \tag{4.20}$$

$$(4u' v v' - 2uv'^2)A + (vv'^2 - 4\varepsilon_3 u'^2 v^2)B = 0. \tag{4.21}$$

First, we suppose that A and B are linearly dependent at some $s \in I$. Then there are constants κ_1 and κ_2 such that $\alpha' - \kappa_1\beta' = \kappa_2\beta$. By using the properties of α and β, we get $u = \varepsilon_3\kappa_1$ and $v = \varepsilon_3\kappa_1^2$, which is a contradiction by the definition of non-vanishing function q. Thus, A and B are linearly independent for all s. From (4.20) and (4.21), we have

$$v'^2 - 4\varepsilon_3 u'^2 v = 0, \tag{4.22}$$

$$u' v(2uu' - \varepsilon_3 v') = 0, \tag{4.23}$$

$$2u' v v' - uv'^2 = 0, \tag{4.24}$$

$$vv'^2 - 4\varepsilon_3 u'^2 v^2 = 0. \tag{4.25}$$

Suppose that the open subset $\mathcal{U} = \{p \in M | u'(p) \neq 0\}$ is not empty. Eq. (4.23) gives

$$v' = 2\varepsilon_3 uu' \quad \text{on } \mathcal{U}, \tag{4.26}$$

which implies from (4.22) that $u^2 = \varepsilon_3 v$ on $\mathcal{U}$. This is also a contradiction. Thus, $\mathcal{U}$ is empty, in other words, $u' = 0$. Furthermore, from (4.22) we also have $v' = 0$. If we take the scalar product with β in the Eq. (4.18)), then we have $\langle \beta'' \times \beta', \beta\rangle = 0$. Hence, there are smooth functions κ_1 and κ_2 such that $\beta = \kappa_1\beta' + \kappa_2\beta''$. It implies that β and β'' are parallel. Also, from $u' = 0$ and $v' = 0$ we get

$$\langle \alpha'', \beta'\rangle = 0, \qquad \langle \alpha'', \alpha'\rangle = 0. \tag{4.27}$$

For the vector fields α', β, β' and α'', we may put

$$\alpha'' = \kappa_1\alpha' + \kappa_2\beta' + \kappa_3\beta$$

for κ_1, κ_2 and κ_3 are smooth functions. Using (4.27), we see that α'' and β are parallel.

On the other hand, by definition the mean curvature vector field H of M is obtained as follows:

$$H = \tfrac{1}{2}\varepsilon_4 q^{-3/2}\langle(\alpha' + t\beta') \times \beta, \alpha'' + t\beta''\rangle.$$

Since β'', α'' and β are parallel to each other, it is easily proved that H vanishes identically. Consequently, by using the classification theorem of a ruled minimal surface in $\mathbb{E}^3_1$ [16] we conclude that the surface of type M_+^1 (resp. M_-^1) are an open part of the helicoid of the 1st kind and the helicoid of the 2nd kind as space-like surface (resp. time-like surface), and the surface of type M_+^3 is an open part of the helicoid of the 3rd kind. The converse is obvious.

Case 2: Let M be a non-cylindrical ruled surface of type M_+^2 or M_-^2. Then, the surface M is parametrized by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\beta,\beta\rangle = 1$, $\langle\alpha',\beta\rangle = 0$, $\langle\alpha',\alpha'\rangle = \varepsilon_1(=\pm 1)$ and β' is null.

It is easy to get the Gauss map G of the surface M as

$$G = \frac{1}{||(\alpha'+t\beta')\times\beta||}(\alpha'+t\beta')\times\beta.$$

We also put functions q and u as before by

$$q = ||x_s||^2 = \varepsilon_4\langle x_s, x_s\rangle, \qquad u = \langle\alpha',\beta'\rangle,$$

which give

$$q = \varepsilon_4(2ut+\varepsilon_1), \qquad G = q^{-1/2}(A+tB), \tag{4.28}$$

where we put $A = \alpha'\times\beta$ and $B = \beta'\times\beta$ and the region of t is chosen so that $q > 0$. The Laplacian Δ of M can be expressed as [14]

$$\Delta = -\varepsilon_4\left(-\frac{1}{2}\frac{1}{q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} + \frac{1}{q}\frac{\partial^2}{\partial s^2}\right) - \left(\frac{1}{2}\frac{1}{q}\frac{\partial q}{\partial t}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\right). \tag{4.29}$$

Using (4.28) and (4.29) we can obtain by a direct computation

$$\begin{aligned}\Delta G = {} & (-2u^2q^{-2} + u''tq^{-2} - 4\varepsilon_4 u'^2t^2q^{-3})G \\ & + q^{-5/2}\{\varepsilon_4 uBq + 3u't(A'+tB') - \varepsilon_4(A''+tB'')q\}.\end{aligned} \tag{4.30}$$

Suppose that M is of pointwise 1-type Gauss map. Similarly to Case 1, using (4.1) and (4.30) we have

$$\varepsilon u\langle B, B''\rangle B = 0, \tag{4.31}$$

$$\begin{aligned}& 2\varepsilon u\langle B,B''\rangle A - 4\varepsilon_4 u^2 B'' + 6\varepsilon_4 uu'B' \\ & \quad + \varepsilon(3u'^2 + 2u\langle A'',B\rangle + 2u\langle A,B''\rangle + \varepsilon_1\langle B,B''\rangle)B = 0,\end{aligned} \tag{4.32}$$

$$\begin{aligned}& 4\varepsilon_4u^2A'' - 6\varepsilon_4uu'A' - \varepsilon(3u'^2 + 2u\langle A'',B\rangle + 2u\langle A,B''\rangle + \varepsilon_1\langle B,B''\rangle)A \\ & \quad + 4\varepsilon_1\varepsilon_4uB'' - 3\varepsilon_1\varepsilon_4u'B' - (4\varepsilon_4u^3 + 2\varepsilon u^3 + 2\varepsilon u\langle A,A''\rangle + \varepsilon\varepsilon_1\langle A'',B\rangle \\ & \quad + \varepsilon\varepsilon_1\langle A,B''\rangle)B = 0,\end{aligned} \tag{4.33}$$

$$\begin{aligned}& 4\varepsilon_1\varepsilon_4A'' - 3\varepsilon_1\varepsilon_4u'A' - \varepsilon(2u^3 + 2u\langle A,A''\rangle + \varepsilon_1\langle A'',B\rangle + \varepsilon_1\langle A,B''\rangle)A + \varepsilon_4B'' \\ & \quad - (4\varepsilon_1\varepsilon_4u^2 + \varepsilon\varepsilon_1u^2 + \varepsilon\varepsilon_1\langle A,A''\rangle)B = 0,\end{aligned} \tag{4.34}$$

$$A'' - \varepsilon\varepsilon_1\varepsilon_4(u^2 + \langle A, A''\rangle)A - uB = 0. \tag{4.35}$$

We easily see that B is non-zero everywhere. In fact, if B is zero at some point s, then β' and β are parallel. It contradicts the property of β and β'. Consider a subset $\mathcal{U} = \{p \in M | \langle B, B''\rangle(p) \neq 0\}$. If $\mathcal{U}$ is not empty, from (4.31) we have $u = 0$ on $\mathcal{U}$. And from (4.32) we also get $\langle B, B''\rangle B = 0$ on $\mathcal{U}$, which is a contradiction on $\mathcal{U}$. Therefore, $\mathcal{U}$ must be empty. Thus, we have

$$\langle B, B''\rangle = 0. \tag{4.36}$$

Consequently, substituting (4.36) into (4.31), (4.32), (4.33), (4.34) and (4.35), we can eliminate A'', B'', A' and B' so that

$$2\varepsilon_1 u u'^2 A - u'^2 B = 0. \tag{4.37}$$

We now suppose that A and B are linearly dependent at some $s \in I$. Then, there are constants κ_1 and κ_2 such that $\alpha' - \kappa_1\beta' = \kappa_2\beta$. By using the properties of α and β, we can obtain $\alpha' = \kappa_1\beta'$, which is a contradiction. Thus, A and B are linearly independent for all s. From (4.37) we show that $u' = 0$. Since $\langle B, B''\rangle = 0$ and $\langle B, B'\rangle = 0$, we have $\langle B, B''\rangle = \langle \beta'', \beta''\rangle = 0$, i.e., β'' is light-like or zero. If β'' is light-like, there is a non-zero smooth function κ such that $\beta'' = \kappa\beta'$ by Lemma 2.1. Hence, we have $\beta = F(s)\mathbb{C}$, where $\mathbb{C} = (c_1, c_2, c_3)$ is a constant light-like vector field in $\mathbb{E}^3_1$ and $F(s)$ is a positive smooth function (cf. [14]). However, there is no such vector field β satisfying $\langle \beta, \beta\rangle = 1$. After all, β'' is the zero vector. As in the previous case, if we examine the relationship among α', β, β' and α'' we find that α'' and β are parallel. Similarly to Case 1, the character of α and β makes the mean curvature vector field H vanish everywhere.

Hence, in this case, the surfaces of types M^2_+ (resp. M^2_-) is an open part of the conjugate of Enneper's surfaces of the 2nd kind as space-like surface (resp. time-like surface) according to Theorems 3 and 4 of [16]. Furthermore, the converse also holds. Thus, this completes the proof. □

Theorem 4.3. *Let M be a null scroll with pointwise 1-type Gauss map in a three-dimensional Minkowski space. Then, M is an open part of one of the following surfaces:*

1. *a Minkowski plane,*
2. *a flat B-scroll if B' is light-like,*
3. *a non-flat B-scroll if B' is non-null.*

Proof. Let $\alpha = \alpha(s)$ be a light-like curve in $\mathbb{E}^3_1$ and $B = B(s)$ be a light-like vector field along α. Then, the null scroll M is parametrized by

$$x = x(s, t) = \alpha(s) + tB(s)$$

such that $\langle \alpha', \alpha'\rangle = 0$, $\langle B, B\rangle = 0$ and $\langle \alpha', B\rangle = 1$.

We have the natural frame $\{x_s, x_t\}$ given by

$$x_s = \alpha' + tB', \quad x_t = B. \tag{4.38}$$

Again, we define smooth functions q, u and v as follows:

$$q = ||x_s||^2 = \langle x_s, x_s\rangle, \quad u = \langle \alpha', B'\rangle, \quad v = \langle B', B'\rangle \tag{4.39}$$

Similarly as before, the Laplacian Δ of M can be given as follows [13]:

$$\Delta = -2\frac{\partial^2}{\partial s\partial t} + \frac{\partial q}{\partial t}\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}. \tag{4.40}$$

Furthermore, the Gauss map G is determined by

$$G = \left(\frac{1}{\|x_s \times x_t\|}\right)(x_s \times x_t) = C + tD, \tag{4.41}$$

where we put $C = \alpha' \times B$ and $D = B' \times B$.

Suppose that M is of pointwise 1-type Gauss map. Then, (1.2) together with (4.40) and (4.41) gives

$$2D' + (ft - 2u - 2vt)D + fC = 0. \tag{4.42}$$

Taking the scalar product with C' and D' in (4.42), respectively, we obtain the following equations:

$$v' + fvt - 2v^2t = 0, \tag{4.43}$$

$$2v^2 - fv = 0. \tag{4.44}$$

Consider an open subset $\mathcal{U} = \{p \in M | v(p) \neq 0\}$. We suppose that $\mathcal{U}$ is not empty. Then, on a component $\mathcal{C}$ of $\mathcal{U}$, we have $f = 2v$ by (4.44). Together with (4.43), we see that v is a constant. Consequently, by continuity, $\mathcal{C}$ must be the whole space M. In this case, we have null frame field $\{\alpha', B, C\}$ in $\mathbb{E}^3_1$ satisfying the following conditions:

$$\langle\alpha',\alpha'\rangle = \langle B, B\rangle = 0, \quad \langle\alpha', B\rangle = 1, \quad \langle\alpha', C\rangle = \langle B, C\rangle = 0,$$
$$\langle C, C\rangle = 1, \alpha'' = -u\alpha' + \langle\alpha'', \alpha' \times B\rangle C, \quad B' = uB + \langle\alpha' \times B, B'\rangle C,$$
$$C' = -\langle\alpha' \times B, B'\rangle\alpha' - \langle\alpha'', \alpha \times B\rangle B.$$

Using (1.2) and $f = 2v$, we obtain that $2v = \langle\alpha' \times B, B'\rangle$ is a constant. Thus, M is a B-scroll (cf. [1]).

If v is identically zero, then B' is zero or light-like. Suppose that B' is the zero vector, i.e., B is a constant vector. Then $D = 0$, which gives $\Delta G = 0$. Consequently, M is a Minkowski plane. If B' is light-like, then B' and B are linearly dependent by Lemma 2.1. Thus we have $D = 0$, which implies $\Delta G = 0$. In that case, we see that the mean curvature vector field H vanishes identically and the Gaussian curvature is also zero. Consequently, M is a flat B-scroll (see Example 3.8). This completes the proof. □

Combining the results of Theorems 4.1, 4.2 and 4.3, we have

Theorem 4.4 (Classification). *Let M be a space-like ruled surface in a three-dimensional Minkowski space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following surfaces:*

1. *a Euclidean plane,*
2. *the hyperbolic cylinder,*
3. *the helicoid of the 1st kind,*
4. *the helicoid of the 2nd kind,*
5. *the conjugate of Enneper's surface of the 2nd kind.*

Theorem 4.5 (Classification). *Let M be a time-like ruled surface in a three-dimensional Minkowski space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following surfaces:*

1. *a Minkowski plane,*
2. *the Lorentz circular cylinder,*
3. *the circular cylinder of index* 1,
4. *the helicoid of the* 1*st kind,*
5. *the helicoid of the* 2*nd kind,*
6. *the helicoid of the* 3*rd kind,*
7. *the conjugate of Enneper's surfaces of the* 2*nd kind,*
8. *a flat B-scroll if* B' *is light-like,*
9. *a non-flat B-scroll if* B' *is non-null.*

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